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Solutions of Continuous and Discrete Time Lyapunov Linear Operator Equations & Its Applications

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By

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بسم الله الرحمن الرحيم

﴿ ۞ وَعِندَهُۥ مَفَاتِحُ ٱلۡغَيۡبِ لَا يَعۡلَمُهَآ إِلَّا هُوَۚ وَيَعۡلَمُ مَا فِي
ٱلۡبَرِّ وَٱلۡبَحۡرِۚ وَمَا تَسۡقُطُ مِن وَرَقَةٍ إِلَّا يَعۡلَمُهَا وَلَا حَبَّةٖ فِي
ظُلُمَٰتِ ٱلۡأَرۡضِ وَلَا رَطۡبٖ وَلَا يَابِسٍ إِلَّا فِي كِتَٰبٖ مُّبِينٖ ٥٩ ﴾

صدق الله العلي العظيم

سورة الأنعام – آية (59)

ا لإ هـد ا ء

إلى كل الذين ساهموا بعملهم وكرمهم ويحبهم بجعل الرسالة تخرج إلى النور لهم كل الشكر والثناء والعرفان

عـا مـر

Abstract

The main theme of this work can divided in main aspects:

First, We modify some theorems to ensure: the existence and uniqueness of the solution for the Lyapunov and Quasi - Lyapunov operator equations. As well as we study and discuss the existence . and uniqueness of the solution of the discrete-time, Sylvester and Lyapounv operator equations.

Second, the range of he Quasi - Lyapunov equation is studied and , we study the nature Discrete - tithe as well as, the study of the range τ_{AB} and τ_A are introduce.

Third the nature of the solution for the Lyapunov and Quasi - Lyapunov operators (Continuous - time) are studied for special - types of operators.

Also, we study the nature solution and the range of the generalization continuous - tithe Lyapuriov operator equations.

TABLE OF NOTATIONS

IR The field of real numbers

C The field of complex numbers

H Infinite dimensional complex - separable Hilbert space

B (H) The Ban ach algebra of all bounded linear Operators On a Hilbert space H.

σ (A) The spectrum of the operator A.

Range (A). The range of the operator A.

< , > Inner product.

||X|| Norm of X.

A* The adjoint of the operator.

F   The filed

R   The ring

ker (x) The kernel of The operator X

A^{-1} The inverse of the operator A

{Xn} The sequence of vector .

I   Identity operator .

Introduction :

Thu Lyaptinon operator equation are of two types . The first type is continuous - time Lyapunov operator equation which takes the from $A^*X + XA = W$, where A and W are known operators defined on a Hilbert space H , X is the known operator that must be determined , [3] , and [5] .

The second type is discrete - time Lyapunoy operator equation which takes $A^*XA – X = W$, where A and W are known operators defined on a Hilbert space H , and X is the unknown operators. that must be determined , [5].

This work concern with special types of the linear operator equation namely the Qusai - Lyapunov operator equation.

These types of linear operator equations have many real life applications in physics biotechnology , [3] and [4] . This work is study. of the nature of solution foe the linear Lyapunov, and Qusai - Lyapunov operator equations.

This thesis consist of five chapters in chapter one , we recall some definitions basic concept and some properties which are important for the discussion of our later results

In Chapter two we give some modifications for Selvester - Rosenblum theorem to guarantee the existence and uniqueness foe the solution of discrete - time Lyapunov equation - as well as the natural of the solution for this equation operators studied .

In Chapter three we study the nature of the solution of the continuous – time Lyapunov and Qnsai - Lyapunov operator equations for special types of operators as well as the study of the range of τ_A , τ_{AB} and μ_A .

where :

$\tau_A (X) = A^* X + XA$.

$\ell_A(X) = AX + X^*A$

$\mu_A (X) = AX + XA$

X ⌡B(H) and A is a fixed operator in H .

In Chapter four , we discuss the nature of solution for generalization of continuous - time Lyapunove operator equations.

In Chapter five , we study and discuss the range of the generalization of continuous - time Lyaponove operator equations.

Chapter One

Basic Operators Concepts

CHAPTER ONE

BASIC OPERATORS CONCEPTS

(1.1) Basic Definitions:

In this section, we give some definitions which are important for the .study of our later results.

Definition (1.1.1), [11] :

Let V be a vector space real or complex then V is called a normed space if there is a norm function ||.|| : V → R defined on it, and if this space is complete with respect to this norm, then it is called a Banach space, thus a Banach is a complete normed vector space.

Definition (1.1.2) [11]:

A Banach space V is called a Banach algebra if there is a multiplication <u, v> = uv :VxV which is linear each factor in particular is a ring (not necessary communicative) and ||uv|| ≤ || ||v|| all u , v in V .

<u>Definition (1.1.3) [11]:</u>

The space V is called an inner product space if there is an inner function <…> : V x V → R or C defined on it. If this space is complete with respect to the norm induced by this inner product, then it is called a Hilbert space.

<u>Definition (1.1.4) , [10] :</u>

Let X and Y be vector spaces. A map B:X → Y is called a linear operator (map) if

$B(\lambda x + \mu z) = \lambda Bx + \mu Bz \quad , \forall\ x , z \rfloor X$ and $V\lambda , \mu \rfloor F$.

<u>Definitions (1.1.5) , [5]:</u>

An equation of the form

L(X) = W, …………………………… (1-1)

is said to be an operator equation, where L and W are known operators defined on a Hilbert space H, and X is the unknown operator that must be determined.

In equation (1-1), if the operator L is linear then this equation is said to be linear operator equation. Otherwise, it is a non-linear operator equation.

Definition (1.1,6), [7] :

An operator A on a Hilbert space H is said to be self - adjoint if $A^* = A$.

Definition (1.1.7) [7]:

An operator . A on Hilbert space H is said to be a skew – adjoint if $A^* = - A$.

Definition (1.1.8), [7]:

An operator A on a Hilbert space H is said to be normal if $A^*A=AA^*$. That is, $\langle A^*Ax, x\rangle = \langle AA^*x, x\rangle$ for all x in H.

Definition (1.1.9), [7]:

An operator A on a Hilbert space H is said to be hyponormal if $A^*A - AA^* \geq 0$. i.e.,

$\langle (A^*A - AA^*) x, x\rangle \geq 0 \quad \forall x \rfloor H$.

Definition (1.1.10), [7]:

An operator A on a Hithert space H is said to be * - paranormal if

$\|A^2x\| \geq \| A^*x \|$, for every unit vector x in H. Equivalently A is * - paranormal if $\|A^2x \| \, \|x \| \geq \|A^*x\|^2$ for every x in H.

Definition (1.1.11), [7] :

An operator A on a Hilbert space is said to be binormal if A*A commutive with AA*. i.e.,

A*AAA* = AA*A*A .

Definition (1.1.12), [7] :

An operator on a Hilbert space H is said to be quasinormal if A commutive A*A. i.e.,

AA*A = A*AA,

Definition (1.1.13), [7]:

An operator A on a Hilbert space H is called Θ - operator if A*A commutive with (A + A*). i.e.,

A*A (A + A*) = (A + A*) A A*.

Definition (1.1.14), [17]:

If B(H) is a Banach algebra with identity and A ⌋ B(H), the spectrum of a, denoted by σ (A), is defined by:

σ (A) = { σ ⌋ : A - αI is not invertible }.

The left spectrum , σ_ℓ (A) , is the set

{α ⌋ F: A – αI is not left invertible}, the right spectrum

σ_r (A) , is the set {α ⌋ F : A - αI is not right invertible }.

<u>Definition (1.1.15), [15]:</u>

Let X be a Banach space over C , and let A ⌋ B(H), defined

σ_π (A) = {λ ⌋ C : A - λI is not bounded below}

σ_π (A) is called the approximate point spectrum of A. An important subset of

σ_π (A) is the point spectrum or eigenvalues of A, σ_p(A)

where: σ_p (A)= {λ ⌋ C : ker (A - λI) ∨ {0}}

Also, defined σ_δ (A) = {λ ⌋ C : ker (A - λI is not *surjecrive*}, σ_δ (A) is called the defect spectrum of A .

<u>Definition (L1.16), [7]:</u>

An operator A on a Hilbert space His said to be compact if, given any sequence of :vectors {x,}. such that ||Xn|| is bounded, {AXn}, has a convergent subsequence.

<u>Definition (1.1.17), [2]:</u>

A linear mapping τ from a ring R to itself is called a derivation, if τ (ab) = a τ (b) + τ (a)b for all a, b in R.

Definition (1.1.18), [2]:

Let R be a ring, a Jordan derivation f : R → R is defined to be an additive mapping satisfying $f(a^2)$= af(a) + f(a)a . Now, let R be *- ring, i.e., a ring with involution *.

Definition (1.1.19), [2]1:

A linear mapping T : R → R is called Jordan *- derivation, if for all

a, b ⌋ R and $\tau(a^2)$ = a τ + τ (a)a*. If R is a ring with the trivial involution, a* = a, then the set of all Jordan derivation is equal to set of all Jordan derivations.

Definition (1.1.20), [7]:

An operator A on a Hilbert space H is said to be isometric if A*A = I, that is ||Ax|| = ||x|| , for all x in a Hilbert space H.

(1.2) Basic Properties and Theorems :

In this section, we give examples and elementary properties and theorems of operator.

Proposition (1.2.1), [9]1:

1. If A is a Heimetian (self - adjoint) operator that is, A = A*, then A is normal.

2. If U is a unitary operator, that is, U*U= UU* = I, then U is normal.

3. If A is a skew - Hermition (skew - adjoint), that is, A* = - A, then A is normal .

Note that, the converses of statements (1), (2), and (3)

Example (1.2.1):

Let I : H → H be the identity operator on a Hilbert space H., and A=2iI. Therefore, A* = -2iI Hence, A*A =AA* . This is normal operator, but it is not self - adjoint, since A ∨ A*. Also, A is not unitary operator because A*A = AA* ∨ I.

Example (1.2.2):

As for (3) in proposition (1.2.1), let

A = $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ one can get A* = $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$

Hence A*A = AA* and A is normal, but A is not a skew - adjoint

Proposition (1.2.2), [1]:

Let A be an operator on a Hilbert space H, then the following statements are equivalent:

1. A is normal.

2. ||Ax|| = ||A*x|| for all x in a Hilbert space H.

Proposition (1.2.3), [8]:

1. A is normal operator if and only if A - λI is normal for each λ in C .

2. If A is normal and A^{-1} exists, then A^{-1} is normal.

3. Let A and B be normal operators, then AB is normal if AB* = B*A.

4. Let A and B be normal operators, then A + B is normal if AB* = B* A .

Theorem (1.2.1), [8]:

If $\sigma(A) \cap (-A) = \Phi$, then A and A^2 commute with exactly the same operators.

Remark (1.2.1),[8]:

Note that if $\sigma(A) \cap \sigma(-A) \vee \Phi$, then theorem (1.2.1) may not hold.

Example (1.2.3):

Let A= $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and B= $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. It is easy checked that:

$A^2B = BA^2$, , (A) = {1 , 1} and σ (-A) = {-1, 1}, therefore

$\sigma(A) \cap \sigma(-A) \vee \Phi$, but A does not commute with B.

Now we give the following corollary that was mentioned as remark (1.2.1).

Corollary(1.2.1),[8]:

If A^2 is normal and $\sigma(A) \cap \sigma(-A) = \Phi$, then A is normal.

Proof:

Since A^2 is normal and commutes with A, then $A^*A^2 = A^2A^*$. Since $\sigma(A) \cap \sigma(-A) = \Phi$, then by theorem (1.2.1), $A^*A = AA^*$ and A is normal.

Proposition (1.2.4),[12]:

Let A be an operator on a Hilbert space H. Then, the following statements are equivalent:

1. A is hyponormal.
2. $\|A^*x\| \leq \|Ax\| \ \forall \ x \in H$.

Proposition (1.2.5), [12]:

Let A be a hyponomial operator, then :

1- A - λI is hyponormal for all λ in C .

2. If A^{-1} exists, then A^{-1} is hyponormal.

Proposition (1.2.6), [12]:

Every nonzero hyponormal operator compact operator is normal.

Remark (1.2.2),[12]:

Every hyponbrmal operator is * - paranormal. In particular, every normal operator is * - paranormal.

Proof:

Let T be a hyponormal operator and let x in a Hilbert space H such that $\|X\| = 1$, then $\|T^*x\|^2 \leq \|Tx\|^2$. Also $\|T^*x\|^2 \leq \| T^2X\|$.

$\|T^*x\|^2 \leq \|Tx\|^2 \leq \|T^2x\|$. Hence T is a* - paranormal operator.

Remark (1.2.3),[8],:

1. Every normal operator is binomial.

2. Every quasi normal operator is binormal.

Proposition (1.2.7) , [8]:

1. A is binomial if and only if A* is binomial.
2. If A is binormal and a is any complex scalar, then αA is binormal.
3. If A is binomial, then A^{-1} is binonnal if A^{-1} exists.

Proposition (1.2.8), [12]:

Let A be hyponormal and binomial operator , then A^n is hypononnal for n ≥

Proposition (1.2.9), [8]:

Let A be an operator on Hilbert Space H , then the following statements a equivalent:

1. A is a Θ - operator.

2. $A^*(A^*A - AA^*) = (A^*A - A^*A) A$.

<u>Proof:</u>

Let A be a Θ - operator, then $A^*(A^*A - AA^*) = A^*A^*A - A^*AA^*$.

By Θ - operator, we get $A^*AA - AA^*A = (A^*A - AA^*)A$. Conversely, since $A^*(A^*A — AA^*) = (A^*A — AA^*)A$, then

$A^*A(A+A^*) = A^*AA + A^*AA^* = A^*A^*A + AA^*A = (A^*+A)(A^*A)$ Hence A is a Θ - operator.

<u>Proposition (1.2.10), [8]:</u>

If A is a Θ - operator and A^{-1} exists, then A^{-1} is a Θ - operator.

<u>Remark (1.2.4):</u>

1. Every normal operator is a Θ - operator is a Θ - operator.
2. Every quasinormal operator is a Θ - operator .

<u>Proof</u> :

1. Let A be a normal operator, that is (A*A = AA*), then

(A*A) (A + A*)= A*AA + A*AA*,

=AA*A + A*A*A,

= (A + A*)AA*. Thus

A is Ɵ - operator .

2. Let A be a quasi noromal operator, then

A(A*A) = (A*A) A . Therefore

(A*A)A* = A*(A*A). thus

(A*A) (A+A*) = (A*A)A + (A*A)A*,

= A(A*A) + A*(A*A),

¯(A*+A) (A*A).

Hence A is a Ɵ - operator .

Proposition (1.2.11),[7]:

Let A and B be operators on a Hilbert space H, then:

I. If A is compact Operator and V is any operator then AB is compact.

2. If A is compact operator and a is any scalar then αA is compact.

3. If A and B are compact operators then A + B is compact.

Chapter Two

Solution of Discrete – Time Operator Equations

CHAPTER TWO

Solution of Discrete - Time Operator Equations

(2.1) Some Types of Operator Equations:

(1) Continuous and discrete - time Sylvester operator equations:

$$AX \pm XB = \alpha C, \qquad (2.1)$$

$$AXB \pm X = \alpha C. \qquad (2.2)$$

(2) Continuous and discrete - time Lyapunnov operator equations:

$$A^*X - XA = \alpha C, \qquad (2.3)$$

$$A^*XA - X = \alpha C \qquad (2.4)$$

Where A, B and C are given operators defined on a Hilbert space H , X is an operator that must be determined, $\propto$ ix any scalar, and A* is the adjoint of A,[4]

In general, these operator equations may have one solution, infinitive set of solutions or no solution.

In this section, existence and uniqueness of the solution of eq. s (2.2) and (2.4), when B is an invertible operator in eq. (2.2), and A is an invertible operator in eq. (2.4) are studied.

The discrete - time Sylvester equation can be transformed into continuous - time Sylvester operator equation as follows:

Multiply eq. (2.2) from the right by , then eq. (2.2) becomes:

$$AXBB^{-1} \pm XB^{-1} = \alpha\, CB^{-1}$$

$$AX \pm XB^{-1} = \alpha\, CB^{-1}$$

Let $CB^{-I} = W$, the above equation becomes :

$$AX \pm XB^{-1} = \alpha W \qquad (2.5)$$

Also, the discrete - time Lyapunov operator equation can be transform to continuous - time Lyapunov operator equation as follows:

Multiply eq. (2.4) from the right by A^{-1}, then eq. (2.4) becomes :

$$A^*XAA^{-1} - XA^{-1} = \alpha\, CA^{-1} \qquad (2.6)$$

Let $CA^{-1} = W$, then eq. (2.6) becomes :

$$A^*X - XA^{-1} = \alpha\, W. \qquad (2.7)$$

Recall that, the spectrum of the operator $A \equiv \sigma(A)\ \{\lambda \rfloor C : (A - \lambda I)$ is not invertible}, and B(H) is the Banach space of all bounded linear operators defined on- the Hilbert space H , [5] .

Sylvester - Rosenblum Theorem(2.2.1),[5].

If A and W are operators in B(H) (panach algebra of all bounded linear, operators defined on a Hilbert space H) , such that

σ (A) $\cap$ (B) = ø , then the. operator equation AX - XB = α C

(continuous - time Sylvester operator equation) has a unique solution X, for every operator C.

The following corollaries give the unique solution for the operator eq. (2.5).

Corollary (2.2.1):

If A and B are operators in B(H) , and B^{-1} exists such that

$\sigma(A) \cap \sigma(B^{-1}) = ø$,

then, the operator equation $AX - XB^{-1} = \alpha W$, has a unique solution X for every operator W ⌋ B(H).

Corollary (2.2.2):

If A and B are operators in B(H) , and B^{-1} exists, such that

$\sigma(A) \cap \sigma(-B^{-1}) = ø$, then the operator equation $AX + XB^{-1} = \alpha W$, has a unique solution for every operator W ⌋ B(H) .

Proposition (2.2.1):

Consider eq. (2.5) , σ (A) ∩ σ (B^1) = ø then the operator

$\begin{bmatrix} A & -\alpha W \\ 0 & B^{-1} \end{bmatrix}$ is defined on $H_1 \oplus H_2$ is similar to the operator $\begin{bmatrix} A & 0 \\ 0 & -B^{-1} \end{bmatrix}$.

Proof:

Since σ (A) ∩ σ (B^{-1}) = ø , then by Sylvester - Rosenblum theorem , the operator equation AX – XB^{-1} = αW, has a unique solution X. Also,

$$\begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} A & o \\ 0 & B^{-1} \end{bmatrix} = \begin{bmatrix} A & -\alpha W \\ 0 & B^{-1} \end{bmatrix} \begin{bmatrix} I & X \\ 0 & I \end{bmatrix}$$

But $\begin{bmatrix} I & X \\ 0 & I \end{bmatrix}$ is similar to the operator , so $\begin{bmatrix} A & 0 \\ 0 & -B^{-1} \end{bmatrix}$ is similar to $\begin{bmatrix} A & -\alpha W \\ 0 & B^{-1} \end{bmatrix}$.

Corollary: (2.2.3)

Consider eq. (2.5), σ (A) ∩ σ (-B^{-1}) = ø , then

The operator $\begin{bmatrix} A & -\alpha W \\ 0 & -B^{-1} \end{bmatrix}$ is defined on $H_1 \oplus H_2$ is

Similar to the operator $\begin{bmatrix} A & 0 \\ 0 & -B^{-1} \end{bmatrix}$

Notes:

(1) The converse of the above proposition is not true in general.

(2) If the condition $\sigma(A) \cap \sigma(B^{-1}) = \emptyset$, fails to satisfied then the operator equation $AX - XB^{-1} = \alpha W$, may have no solution.

(3) If the 'condition $\sigma(A) \cap (-B^{-1}) = \emptyset$ **,** fails to satisfy then

The operator equation $AX + XB^{-1} = \alpha W$, may have no. solution.

Now, the following Corollary gives the unique - solution the operator eq. (2.7)..

<u>Corollary: (2.2.4)</u>

If A an operator in B(H) , A^{-1} exists such that

$\sigma(A^*) \cap \sigma(A^{-1}) = \emptyset$, then eq. (2.7) has a unique solution X, for every operator W.

<u>Proposition: (2.2.2)</u>

Consider eq. (2.7), if $(A^*) \cap (A^{-1}) = \emptyset$, then the

Operator $\begin{bmatrix} A^* & -\alpha W \\ 0 & A^{-1} \end{bmatrix}$ is defined on $H_1 \oplus H_2$ is similar to

The operator $\begin{bmatrix} A^* & 0 \\ 0 & A^{-1} \end{bmatrix}$

Proof:

Since σ (A*) ∩ σ (A^{-1})= ø , then by Sylvester — Rosenblum theorem, eq. (2.7) has a unique solution. Also

$$\begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} A^* & 0 \\ 0 & A^{-1} \end{bmatrix} = \begin{bmatrix} A* & -\alpha W \\ 0 & A^{-1} \end{bmatrix} \begin{bmatrix} I & X \\ 0 & I \end{bmatrix}$$

But $\begin{bmatrix} I & X \\ 0 & I \end{bmatrix}$ is invertible , so $\begin{bmatrix} A* & 0 \\ 0 & A^{-1} \end{bmatrix}$ is

Similar to $\begin{bmatrix} A* & -\alpha W \\ 0 & A^{-1} \end{bmatrix}$.

Notes:

(1) The converse of the above proposition is not true in general.

(2) If the condition $\sigma(A) \cap \sigma(A^{-1}) = ø$, fails to Satisfied then eq. (2.7) may have one solution; an infinite number of solutions or it may have no solution.

(2.3) The Nature of The Solution For The Discrete - Time Lyapunov Operator Equations

In this section , we study the nature of the solution for special types of the linear operator equation , namely the discrete - time Lyapunov equation.

Proposition: (2.3.1) , [2] :

If A is a normal operator and .A^{-1} exists, then A^{-1} is normal .

Proposition: (2.3.2) , [2] :

If A is a hyponormal operator, and A^{-1} exists, then A^{-1} is hyponormal.

Remarks: (2.3.1)

(1) If A , A^{-1} , and W are self adjoint operators , the eq. (2.7) , may or may not have a solution. Moreover, if it has a solution then it may be non self - adjoint.

This remark can be easily observed in matrices.

(2) Consider eq. (2.7) , if W has self - adjoint operator, then it is not necessarily that X = X* .

(3) If A , A^{-1}, and W are skew - adjoint operators, then eq. (2.7) has no solution.

Proposition: (2.3.5) , [2]

(1) If A is a self - adjoint operator , then A is normal.

(2) If A is skew - adjoint operator , then A is noimal.

Remark: (2.3.2)

Consider eq. (2.7) ,

(1) If A and W are normal operators , then the solution X is not necessarily normal operator.

(2) If W is noimal operator and A is any operator , then it is not necessarily that the solution X is normal operator.

Remark: (2.3.3)

(1) Consider eq. (2.7) , if W is compact operator , then A , A^{-1} , and X are not necessarily compact operators.

(2) If A or W or A^{-1} compact operator, and the solution of eq. (2.7) exists, then it is not necessarily to be compact.

<u>Putnam - Fugled Theorem: (2.3.1) , [1]</u>

Assume that M, N, T ⌋ B(H) , where M and N are normal.

if MT = TN then M*T = TN*

<u>Definition:(2.3.1), [1] :</u>

An operator M is said to be dominant if

||(T - z) * x || (T – z) x || , for all z ⌋ σ (T) and x ⌋ R

<u>Definition: (2.3.2) [1] :</u>

An operator M is called M - hyponormal operator if

|| (T — z)*x || ≤ ||M ||(T - z)x || , for z ⌋ C and x ⌋ H.

<u>Theorem: (2.3.2) , [3]</u> :

Let M be dominant operator and N* is M - hyponormal operator.

Assume that MT = IN for some T ⌋ B(H) then M*T = TN*.

<u>Theorem: (2.3.3) , [3] :</u>

Let A and B be two operators that satisfy Putnam - Fugled condition.

The operators equation AX - XB = C has a solution X f and only if

$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ and $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ are similar operator on $H_1 \oplus H_2$.

As corollaries , we have

Corollary : (2.3.1)

If A is normal and A^{-1} exists , then operator equation $A^*X + XA^{-1} = \alpha W$ has a solution if and only if $\begin{bmatrix} A^* & 0 \\ 0 & -A^{-1} \end{bmatrix}$ is similar to $\begin{bmatrix} A^* & -\alpha W \\ 0 & -A^{-1} \end{bmatrix}$.

Corollary : (2.3.2)

If A and B are normal operators and B^{-1} exists then the operator equation $AX - XB^{-1} = \alpha W$ has a solution if and only if $\begin{bmatrix} A & 0 \\ 0 & -B^{-1} \end{bmatrix}$ is similar to $\begin{bmatrix} A^* & -\alpha W \\ 0 & -B^{-1} \end{bmatrix}$.

The following corollaries follows directly from the theorem (2.3.2) .

Corollary : (2.3.3)

If A is a dominant or M – hyponormal operator A^{-1} exists . Then , the operator equation $A^*X + XA^{-1} = \alpha W$

Has a solution if and if $\begin{bmatrix} A^* & 0 \\ 0 & -A^{-1} \end{bmatrix}$ and $\begin{bmatrix} A^* & -\alpha W \\ 0 & -A^{-1} \end{bmatrix}$

are similar operators on $H_1 \oplus H_2$.

Corollary : (2.3.4)

If A and B are dominant or M – hyponormal operator and B^{-1} exists . Then the operator equation $AX - XB^{-1} = \alpha W$ has a solution If $\begin{bmatrix} A & 0 \\ 0 & -B^{-1} \end{bmatrix}$ and $\begin{bmatrix} A & -\alpha W \\ 0 & -B^{-1} \end{bmatrix}$ are similar operators on $H_1 \oplus H_2$.

Proposition :

Consider eq . (2.7) , if A and W are orthogonal operators , and A^{-1} exists and W is also , orthogonal operator , and the solution X of eq. (2.7) is unique then this solution is an orthogonal operator .

Proof :

Consider the operator equation

$A * X + XA^{-1} = W$,

$(A^*X = XA^{-1})^* = W^*$,

Since W is an orthogonal operator ($W^* = W^{-1}$) implies that

$W = (W^{-1})^*$

$X^*A + (A^{-1})^*X^* = W^*$,

$[X^*A + (A^{-1})^*X^* = W^*]^{-1}$

Since A is an orthogonal operator ($A^* = A^{-1}$)

$A^{-1}(X^*)^{-1} + (X^*)^{-1} A^* = (W^*)^{-1}$,

$A^*(X^*)^{-1} + (X^*)^{-1} A^{-1} = W$.

Then $(X^*)^{-1} = X$, so $X^* = X^{-1}$.

Therefore , X is orthogonal operator .

<u>Proposition : (2.3.7)</u>

Consider eq . (2.7) , if A is unitary operator and Q is orthogonal operator and the solution of eq. (2.7) is unique , then this solution is an orthogonal operator .

<u>Proof :</u>

Consider the following linear operator equation

$A^*X + XA^{-1} = W$,

$(A^*X = XA^{-1})^* = W^*$,

$X^*A + (A^{-1})^*X^* = W^*$,

$(X^*A + (A^{-1})^*X^*)^{-1} = (W^*)^{-1}$,

$A^{-1}(X^*)^{-1} + (X^*)^{-1} [(A^{-1})^*]^{-1} = (W^*)^{-1}$

Since A is unitary operator then $A^* = A^{-1}$

So , $A^* (X^*)^{-1} + (X^*)^{-1}A^{-1} = (W^*)^{-1}$,

Since eq. (2.7) has a unique , then

$X = (X^*)^{-1} = (X^{-1})^*$. Therefore , $X^* = X^{-1}$

implies that X is a orthogonal operator .

Then $(X^*)^{-1} = X$, So $X^* = X^{-1}$.

Therefore , X is orthogonal operator .

Proposition : (2.3.7)

Consider eq. (2.7) , if A is unitary operator and W is orthogonal operator and the solution of eq. (2.7) is unique , then this solution is an orthogonal operator .

Proof :

Consider the following linear operator equation

$A^*X + XA^{-1} = W$,

$(A^*X + XA^{-1})^* = W^*$,

$X^*A + (A^{-1})^*X^* = W^*$,

$(X^*A + (A^{-1})^*X^*)^{-1} = (W^*)^{-1}$,

$A^{-1} (X^*)^{-1} + (X^*)^{-1}[(A^{-1})^*]^{-1} = (W^*)^{-1}$

Since A is unitary operator then $A^* = A^{-1}$

So , $A^*(X^*)^{-1} + (X^*)^{-1}A^{-1} = (W^*)^{-1}$,

Since eq. (2.7) has a unique , then

$X = (X^*)^{-1} = (X^{-1})^*$. Therefore , $X^* = X^{-1}$

Implies that X is a orthogonal operator .

Definition : (2.4.1) , [2]

Let R be a ring . A linear (additive) mapping τ from R to R is called a derivation , if

$\tau(ab) = a\,\tau(b) + \tau(a)\,b$, for all a , b in R .

Proposition : (2.4.1)

The map $\tau_{A,B}(X) = AX - XB^{-1}$ is a linear map .

Proof :

Since $\tau_{A,B}(\alpha X_1 + \beta X_2)$

$$\begin{aligned} \tau_{A,B}(\alpha X_1 + \beta X_2) &= A(\alpha X_1 + \beta X_2) - (\alpha X_1 + \beta X_2)\, B^{-1} , \\ &= \alpha AX_1 + \beta AX_2 - \alpha X_1B^{-1} - \beta X_2\, B^{-1} , \\ &= \alpha AX_1 - \alpha X_1B^{-1} + \beta AX_2 - \beta X_2B^{-1} , \\ &= \alpha(AX_1 - X_1B^{-1}) + \beta(AX_2 - X_2B^{-1}) , \\ &= \alpha\, \tau_{A,B}(X_1) + \beta\, \tau_{A,B}(X_2) . \end{aligned}$$

Then $\tau_{A,B}$ is a linear map .

Proposition : (2.4.2)

The map $\tau_{A,B}(X) = AX - XB^{-1}$ is bounded .

Proof :

Since $\| \tau_{A,B} \| = \|AX - XB^{-1}\| \le \|AX\| + \|XB^{-1}\|$

$$\le \|X\|[\|A\| + \|B^{-1}\|] .$$

But A , B^{-1} ⌡ B (H) , $\| \tau_{A,B} \| \le M \|X\|$, where $M = (\|A\| + \|B^{-1}\|)$.

So $\tau_{A,B}$ is bounded .

The following remark shows that the mapping $\tau_{A,B}$ is not derivation .

Remark : (2.4.1)

Since $\tau_{A,B}$ (XY) = A(XY) – (XY)B^{-1} for X, Y $\in$ B (H)

and X $\tau_{A,B}$ (Y) = XAY – XYB^{-1} . Also ,

$\tau_{A,B}$ (X) Y = AXY – XB^{-1} Y then one can deduce that

$\tau_{A,B}$ (XY) $\neq$ $\tau_{A,B}$ (Y) + $\tau_{A,B}$ (X) Y .

Definition : (2.4.2) , [2]

Let R be * - ring , i .e . a ring with involution * . The linear mapping τ from R to R is called Jordan * - derivation ,

if for all a , b $\in$ R ,

$\tau (a^2) = a\, \tau (a) + \tau (a)\, a^*$

Remark : (2.4.3)

The mapping τ:B (H) → B (H) defined by

τ (X) = $\tau_{A,B}$(X) = AX – XB^{-1} is not Jordan * - derivation .

Now , we have the following simple proposition :

Proposition : (2.4.3)

A Rang $(\tau_{A,B}) = \{\alpha\,(AX - XB^{-1}) : X \in B\,(H)\}$,

$= \{A\,(\alpha X) - (\alpha X)B^{-1} : X \in B\,(H)\}$.

Let $X_1 = \alpha X$, then

α Rang $(\tau_{A,B}) = \{AX_1 - X_1B^{-1} : X_1 \in B\,(H)\}$,

$=$ Rang $\tau_{A,B}$.

Remark : (2.4.2)

in general Rang $(\tau_{A,B})^* \neq$ Rang $(\tau_{A,B})$.

(2.5) The range of τ_A

In this section , we discuss and study the map $\tau_A : B\,(H) \to B\,(H)$, where $\tau\,(X) = \tau_A(X) = A^*X - XA^{-1}$, $X \in B\,(H)$.

Proposition : (2.6.1)

The map $\tau_A = A^*X - XA^{-1}$ is a linear map .

Proof :

Since $\tau_A\,(\alpha X_1 + \beta X_2) = A^*(\alpha X_1 + \beta X_2) - (\alpha X_1 + \beta X_2)\,A^{-1}$,

$= \alpha\,A^*X_1 + \beta A^*X_2 - \alpha X_1A^{-1} - \beta X_2A^{-1}$,

$= \alpha\,(A^*X_1 - X_1A^{-1}) + \beta\,(A^*X_2 - X_2A^{-1})$,

$= \alpha\,\tau_A\,(X_1) + \beta\,\tau_A(X_2)$.

Then τ_A is a linear map .

Proposition : (2.5.2)

The map $\tau_A = A^*X - XA^{-1}$ is bounded .

Proof :

Since $\| \tau_A \| = \|A^*X - XA^{-1}\| \leq \|A^*X\| + \|XA^{-1}\|$,

$$\leq \| X \| [\|A\| + \|A^{-1}\|] .$$

But A , $A^{-1} \in B(H)$, let $M = \|A\| + \|A^{-1}\|$, then

$\| \tau_A \| \leq M \| X \|$. So τ_A is bounded .

Remark : (2.5.1)

$\tau_A(XY) = A^*(XY) - (XY)A^{-1}$, for all X , Y $\in B(H)$,

and $X \tau_A (Y) = XA^*Y - XYA^{-1}$.

Also , $\tau_A (X)Y = A^*XY - XA^{-1}Y$. Then one can deduce that

$\tau_A(XY) \neq X \tau_A(Y) + \tau_A(X)Y$.

Remark : (2.5.2)

The mapping τ_A is not Jordan* - derivation .

Now , we have the following simple proposition .

Propositions : (2.5.3)

α Rang (τ_A) = Rang (τ_A) .

Proof :

Since $\supseteq$ Rang $(\tau_A) = \{\alpha (A^*X - XA^{-1}) : X \in B(H)\}$,

$$= \{A^*(\alpha X) - (\alpha X) A^{-1} : X \in B(H)\} .$$

Let $X_1 = \alpha X$, then

α Rang $(\tau_A) = \{A^*X_1 - X_1A^{-1} : X_1 \in B(H)\}$ = Rang (τ_A) .

Remark : (2.5.3)

Rang $(\tau_A)^* \neq$ Rang (τ_A) .

Proposition : (2.5.4)

Rang (τ_A) is linear manifold of operators in B (H) .

Proof :

It is known that rang (τ_A(X)) = {W : τ_A (X) = W , X ⌋ B (H) .

(1) 0 ⌋ Rang (τ_A) since X = 0 ⌋ B (H) } and τ_A (0) .

(2) Let W_1 , W_2 ⌋ Rang (τ_A) we must prove $W_1 - W_2$ ⌋ Rang (τ_A) .

Therefore , ∃ X_1 ⌋ B (H) such that τ_A (X_1) = W_1 and ∃ X_2 ⌋ B (H) .

such that $\tau_A(X_2) = W_2$. Thus , $\tau_A (X_1 - X_2) = A^*(X_1 - X_2) - (X_1 - X_2) A^{-1}$

$$= (A^*X_1 - X_1A^{-1}) - (A^*X_2 - X_2A^{-1})$$

$$= \tau_A (X_1) - \tau_A (X_2) .$$

$$= W_1 - W_2 .$$

Then $W_1 - W_2$ ⌋ B (H) such that τ_A ($X_1 - X_2$) = $W_1 - W_2$. So ,

$W_1 - W_2$ ⌋ Rang (τ_A) .

Therefore , Rang (τ_A) is a linear manifold of operators .

Remark : (2.5.4)

If W ⌋ Rang (τ_A) , then so does W* .

Remark : (2.5.5)

If W_1 , W_2 ⌋ Rang (τ_A) , then W_1 , W_2 is not necessarily in Rang (τ_A) .

Chapter Three

Lyapunov and Qusai – Lyapunov Operator Equation

Chapter Three

Lyapunov and Qusai – Lyapunov

Operator Equations

In this section , we give some types of linear operator equations :

(1) A special type of linear operator equations takes the formula ,

$AX - XB = Y$ …………. (3 – 1)

Where A , B and Y are given operators defined on a Hilbert space H , and X is the unknown operator that must be determined . This linear operator equation is said to be the Sylvester operator equation or continuous – time Sylvester equation , [3] , and [5] .

The author in reference [5] discussed the necessary and sufficient conditions for the solvability of this linear equation . Furthermore , he gave equivalent conditions for the solvability of this linear equation for special types of operators A and B .

(2) The linear operator equation of the form

$A^*X + XA = W$, ……………… (3 – 2)

Where A and W are given operators defined on a Hilbert space H , and X is the unknown operator that must be determined . This linear operator equation is called the Lyapunov operator equation , or the continuous – time Lyapunov equation , [3] and [5] .

The author in reference [3] studied the necessary and sufficient conditions for the solvability of this linear operator equation .

(3) A special case of the continuous – time Lyapunov operator equation

$$AX + XA = W , \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3-3)$$

Where A and W are known operators defined on Hilbert space . H , and X is the unknown operator that must determined , [3] and [4] .

(4) The linear operator equation of the from

$$AX + X^*A = W , \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3-4)$$

Where A and B are given operators defined on a Hilbert space H , and X is the unknown operator that must be determined , X^* is the adoint of X . These linear operator equation (3 – 3) and (3 – 4) are called quasi – Lyapunov operator equations or quasi – continuous – time Lyapunov linear operator equations .

<u>(3.2) The Quasi – Continuous – Time Lyapunov Operator Equations :</u>

The continuous – time Lyapunov equations , are much studied because of it's importance in differential equations and control theory , [6] . Therefore , we devote the studying of the quasi – continuous – time Lyapunov operator equations .

Now , dose eq . (3.2) and eq. (3.4) have a solution ?

If yes , is it unique ?

To answer this question , recall the Sylvester – Rosenblum theorem , [5] .

Sylvester – Rosenblum Theorem (3.2.1) :

If A and B are operators in B (H) such that σ (A) $|$ σ (B) = Φ , then eq. (3 – 1) has a unique solution X for every operator Y .

According to the Sylvester – Rosenblum theorem we have the following corollary .

Corollary (3.2.1) :

If A is an operator such that σ (A) $|$ σ (- A) = Φ , then eq. (3 – 3) has a unique X for every operator W .

Proposition (3.2.1) :

Consider eq. (3.3) , if σ (A) $|$ σ (-A) = Φ , then

The operator $\begin{bmatrix} A & -W \\ 0 & -A \end{bmatrix}$ is defined on $H_1 \oplus H_2$ is similar to $\begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix}$.

Proof :

Since σ (A) $|$ σ (-A) = ϕ . Then by Sylvester – Rosenblum theorem eq. (3.3) has a unique solution X , also :

$$\begin{bmatrix} I & X \\ 0 & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix} = \begin{bmatrix} A & -W \\ 0 & -A \end{bmatrix} \begin{bmatrix} I & X \\ 0 & -A \end{bmatrix} .$$

But $\begin{bmatrix} I & X \\ 0 & I \end{bmatrix}$ is invertible so $\begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix}$ is similar to $\begin{bmatrix} A & -W \\ 0 & -A \end{bmatrix}$.

The converse of the above proposition is not true in general as we see in the following example .

Example (3.2.1) :

let H = $\ell_2(\Phi) = \left\{ X = (x_1, x_2, ...) : \sum_{i=1}^{\infty} |xi|^2 < \infty, xi \in C \right\}$

Define A : H $\rightarrow$ H by A(x_1 , x_2 , ...) = (x_1 , 0 ,0 ,...) . Consider eq. (3.3) , Where W = (x_1 , x_2 , ...) = (0 , x_1 , 0 , ...) . Then X = U is a solution of this equation since (AX + XA) (x_1 , x_2 , …) = (AU + UA) (x_1 , x_2 , …)

A (0 , x_1 , x_2 , …) + U (x_1 , 0 , 0 , …) + (0 , x_1 , 0 , …) = WX

On the other hand , U is solution of eq . (3-3) and

$$\begin{bmatrix} I & U \\ 0 & I \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix} = \begin{bmatrix} A & -W \\ 0 & -A \end{bmatrix} \begin{bmatrix} I & U \\ 0 & -A \end{bmatrix} .$$

Therefore , $\begin{bmatrix} A & -W \\ 0 & -A \end{bmatrix}$ is similar to $\begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix}$.

Moreover 0 is an eigenvalue of A and X = (0 , x_2 , …) is the associated eigenvector .

Therefore , $0 \rfloor \sigma (A) \mid \sigma (-A)$ and hence $0 \rfloor \sigma (A) \mid \sigma (-A) \neq \Phi$.

(3.3) The Nature of the Solution for the quasi – continuous – time Lyapunov operator equations :

In this section , we study the name of the solution of eq. (3 – 3) for special types of operators .

Remark (3.3.1) :

if W is self – ad joint operator , and A is any operator , then eq. (3 -3) may of may not have solution . Moreover , if it has a solution then it may be non self – adjoint .

This remark can easily be checked in matrices .

next if A and W are self – adjoint solution for eq. (3-3) ?

The following theorem gives one such conditions .

Theorem (3.3.1) :

Let A and W be positive self – adjoint operators .

If $0 \notin \sigma(A)$, then the solution X of eq. (3-3) is self – adjoint .

Proof :

Since $0 \notin \sigma(A)$, then it is easy to see that $\sigma(A) \cap \sigma(-A) = \Phi$ and hence eq. (3-3) has a unique solution X by Sylvester – Rosenblum theorem . Moreover ,

$(AX + XA)^* = W^*$,

$A^*X^* + X^*A^* = W^*$,

Since A and W are self – adjoint operators , then $AX^* + X^*A = W$.

Therefore , X^* is also a solution of eq. (3 – 3) . By the uniqueness of the solution one gets $X = X^*$.

Proposition (3.3.1) :

If A and W are self – adjoint operators , and the solution of the equation $AX + X^* A = W$ exists , then this solution X is a unique .

Proof :

Consider eq. (3-4) ,

$$AX\ X^*A = W$$

Since W is self – adjoint operator ,

$$(AX + X^*A)^* = W^* ,$$

$$A^*(X^*)^* + X^*A^* = W^* ,$$

Since A and W is self – adjoint operator ,

$AX + X^*A = W$, since the solution exists , then X is a unique .

The following proposition shows that if the operators A and W are skew – adjoint , and the solution of eq. (3-4) exists then this solution is unique .

Proposition (3.3.2) :

If A and W are skew – adjoint operators , and the solution of eq/ (3-4) exists , then the solution X is a unigue .

Proof :

Consider eq . (3-4)

$$AX + X^*A = W ,$$

Since W is a skew – adjoint operator , so

$$- (AX + X^*A)^* = - W^* ,$$

$$- (A^*(X^*)^* + X^*A^*) = - W^* ,$$

$$(-A^*)X + X^*(-A^*) = - W^* ,$$

Since A and W are skew – adjoint operators , then

$$AX + X^*A = W ,$$

Since the solution X exists , then the solution X is a unique .

Remark (3.2.2) :

If A is a self – adjoint operator and W is a skew – adjoint . Then the solution X of eq . (3-4) is not necessarily exists .

Remark (3.3.3) :

If W is a self – adjoint operator , and A is any operator , then the solution X of eq . (3 -4) is not necessarily self – adjoint operator .

The following example explain this remark .

Example (3.3.1) :

Consider eq. (3-4) , take W = W* = $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, and A = $\begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}$,

$$AX + X^*A = W ,$$

After simple computations one can gets

$$x = \begin{bmatrix} \alpha & 0 \\ 1 & \frac{1}{2} \end{bmatrix} \neq X^* ,$$

Where α is any scalar .

Remark (3.3.4) :

If W is a skew – adjoint and A is any operator , then the solution X of eq. (3-4) is not necessarily exists .

The following example explain this remark .

Example (3.3.2) :

Consider eq . (3-4) , take W = $\begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$ and A = $\begin{bmatrix} 2 & 0 \\ 3 & 0 \end{bmatrix}$

$$AX + X^*A = W ,$$

After simple computations one can gets

$x_2 = 1$ and $x_2 = 0$ which has no solution .

Proposition (3.3.3) , [13] :

If A and W are skew – adjoint operators and eq. (3-2) has only one solution then this solution is also a skew – adjoint .

Proof :

Since $A^* = - A$ and $W^* = - W$ then it easy that to check

$A^*(- X^*) + (- X^*) A = W$ and since the equation has only one solution then $X^* = - X$.

Remark (3.3.7) , [13] :

Consider eq. (3-2) , where the solution of it exists . If A and W are normal operators then this solution is not necessarily normal .

This fact can be seen in the following example .

Example (3.3.4) :

let $H = \ell_2(C)$, consider eq. (3-2) , where $A = iI$ and $W = 0$. Therefore , $- iIX + iIX = 0$. It is easy to check the unilateral shift operator defined by :

$U (X_1 , X_2 , \ldots) = (0 , X_1 , X_2 , \ldots) , \quad \forall (X_1 , X_2 , \ldots) \in \ell_2(C)$

is a solution of the above equation which is non normal operator .

Putnam – Fugled Theorem (3.3.2) , [5] :

Assume that $M , N , T \in B (H)$, Where M and N are normal . If $MT = TN$ then $M^*T = TN^*$.

Recall that an operator T . is said to be dominant if

||(T – Z)*X|| ≤ M || (T – Z) x|| for all z ⌋ σ (T) and x ⌋ H . On the other hand , operator T is called M – hyponormal operator if

||(T – Z)* X|| ≤ M ||(T – Z) x|| for all Z ⌋ C and x ⌋ H , [1] .

In [ ] , the above theorem was generalized as following .

<u>Theorem (3.3.3) , [5] :</u>

Let M be a dominant operator and N* is an M – hyponormal operator . Assume MT = TN for some T ⌋ B (H) then M*T = TN* .

A corollaries , we have .

<u>Corollary (3.3.1) , [13] :</u>

If A is normal operator then eq. (3-2) has a solution if and only if

$\begin{bmatrix} A^* & 0 \\ 0 & -A \end{bmatrix}$ is similar to $\begin{bmatrix} A^* & -W \\ 0 & -A \end{bmatrix}$.

Corollary (3.3.2) , [13] :

If A is dominant or a M – hyponormal operator then the operator equation defined by eq. (3-2) has a solution if and only if

$\begin{bmatrix} A^* & 0 \\ 0 & -A \end{bmatrix}$ and $\begin{bmatrix} A^* & -W \\ 0 & -A \end{bmatrix}$ are similar operator $H_1 \oplus H_2$.

Corollary (3.3.3) , [13] :

If A is a dominant or a . M – hyponormal operator then the operator eq. (3-2) has a solution a solution if and only if

$\begin{bmatrix} A & 0 \\ 0 & -A \end{bmatrix}$ and $\begin{bmatrix} A & -W \\ 0 & -A \end{bmatrix}$ are similar operator $H_1 \oplus H_2$.

Remark (3.3.8) , [13] :

If A (or W) is compact and the solution of eq. (3-2) exists then it is not necessarily compact .

As an illustration to this remark , consider the following examples .

Example (3.3.5) :

Consider the equation A*X + XA = A* + A , where A is a compact operator on an infinite dimensional Hilbert space H . It is clear that X = I is a solution of the above operator equation which is not compact .

Example (3.3.6) :

Consider eq. (3-2) , where $W = 0$. It is clear that the zero operator is compact . Given $A = iI$, then $X = I$ is a solution of eq. (3-2) which is not compact .

Proposition (3.3.4) :

If A is a compact operator then the eq. (3-4) is compact .

Proof :

Since A is compact then X^*A is also compact .

Since A is compact then AX is also compact .

Since AX and X^*A are compact then $AX + X^*A$ is compact .

Therefore W is compact .

(3.4) On The Range of τ_A :

In this section , we discuss the infectivity of the map

$\tau_A : B(H) \to B(H)$ and show that in general the map τ_A is not necessary one – to – one .

Define the mapping $\tau : B(H) \to B(H)$ by

$$\tau(X) = \tau_A(X) = A^*X + XA , \ X \rfloor B(H)$$

Where A is a fixed operator in B (H) .

It is clear that the map τ_A is a linear map , infact

$$\tau_A(\alpha X_1 + \beta X_2) = A^*(\alpha X_1 + \beta X_2) + (\alpha X_1 + \beta X_2) A$$

$$= \alpha \, \tau_A(X_1) + \beta \, \tau_A(X_2)$$

Also , the map τ_A is bounded , since

$\| \tau_A \| = \| A^*X + XA \| \leq \| A^*X \| XA \| \leq \| \ \| [\| A^* \| + \| A \|]$

But $A \in B(H)$ and $\| A^* \| = \| A \|$, thus $\| \tau_A (X) \| \leq M \| X \|$,

Where , $M = 2 \|A\|$, so τ_A is bounded .

The following remark shows that the mapping τ_A is not a derivation .

<u>Remark (3.4.1) , [15] :</u>

Since $\tau_A (XY) = A^* (XY) + (XY) A$ for all $X , Y \in B(H)$ and $X\tau_A (Y) = XA^*Y + XYA$. Also , $\tau_A(X) Y = A^*XY + XAY$. Then one can deduce that $\tau_A (XY) \neq X\tau_A (Y) + \tau_A(X)Y$. To prove this , let $H = \ell_2(C)$ and $A = U$, where is a unilateral shift operator . Then $\tau_U(X) = BX + XU$, where B is the bilateral shift operator .

In this case τ_U is not derivation . To see this , consider

$\tau_U(IU) = \tau_U(U) = BU + U^2$ and

$I\tau_U(U) + \tau_U(I)U = \tau_U (U) + \tau_U(I)U$,

$= BU + U^2 + (B + U)U$,

$= 2BU + 2U^2 = 2 (BU + U^2)$.

It is easily to seen that the mapping $\tau : B(H) \rightarrow B(H)$ defined by

$\tau(X) = \tau_A(X) = A^* + XA$, X ⌋ B(H) is not Jordan * - derivation .

To see this see the above example .

Next , we discuss the infectivity of the map τ_A and show that , in general the map $\tau : B(H) \rightarrow B(H)$ is not necessary one – to – one .

<u>Proposition (3.4.1) , [13] :</u>

Consider the map $\tau(X) = \tau_A(X) = A^*X + XA$. If A is a skew – adjoint operator then τ_A is not one – to – one .

<u>Proof :</u>

Since A is a skew – adjoint operator , then .

$\ker(\tau) = \{X$ ⌋ $B(H) : AX = XA\}$. Therefore I ⌋ $\ker(\tau_A)$ and thus τ_A is not one – to – one .

Now , we have the following proposition .

Proposition (3.4.2) , [13] :

1) Range $(\tau_A)^*$ = Range (τ_A) .

2) α range (τ_A) = Range (τ_A) .

Proof :

1) Since Range $(\tau_A)^* = \{X^*A + A^*X^* , X \rfloor B(H)\}$. Then ,

Range $(\tau_A)^* = \{X^*A + A^*X^* , X \rfloor B(H)\}$. where $X_1 = X^*$.

Therefore , Range $(\tau_A)^*$.

2) α Range $(\tau_A) = \{\alpha(A^*X + AX) , X \rfloor B(H)\}$.

$= \{A^* (\alpha X) + (\alpha X)A , X \rfloor B(H)\}$.

Let $X_1 = \alpha X$, then

Range $(\tau_A) = \{A^*X_1 + X_1A , X_1 \rfloor B(H)\}$

= Range (τ_A) .

(3.5) On The Range of ρ_A :

In this section we study and discuss the range of ρ_A , where

$\rho(X) = \rho_A(X) = AX + X^*A , X \rfloor B(H)$.

Where A is a fixed operator in B(H) .

It is clear that map ρ_A is a linear map . Also the , ρ_A is bounded , since

$\| \rho_A \| = \|AX + X^*A\| \leq \|AX\| + \|X^*A\| \leq \|A\| \, \|X\| + \|X^*\| \, \|A\|$

Since $\|X^*\| = \|X\|$.

Therefore , $\|\rho_A\| \leq 2\|A\| \, \|X\|$,

Let $M = 2 \, \|A\| \geq 0$, so $\| \rho_A \| \leq M \, \|X\|$. Then ρ_A is bounded .

The following steps shows that Range $(\rho_A)^* \neq$ Range (ρ_A) ,

Range $(\rho_A)^* = \{(AX + X^*A)^* , X \rfloor B(H)\}$,

$= \{A^*X + X^*A^* , X \rfloor B(H)\}$,

$\neq$ Range (ρ_A) .

Also , α Range $(\rho_A) = \{\alpha \, (AX + X^*A) , X \rfloor B(H)\}$

$= \{A(\alpha X) + (\alpha X)^* A , X \rfloor B(H)\}$

Let $\alpha X = X_1$

α Range $(\rho_A) = \{AX_1 + X_1{}^* A^* , {}_{X1} \rfloor B(H)\}$

= Range (ρ_A) .

The following remark shows the mapping ρ_A is not – a derivation .

Remark (3.5.2) :

Since $\rho_A (XY) = A(XY) + (XY)^* A$

$= A(XY) + Y^*X^*A$,

for all X , Y ⌋ B(H) ,

and $X\rho_A(Y) = X [AX + Y^*A]$,

$= XAY + XY^*A$.

Also , $\rho_A(X)Y = (AX + X^* A)Y$,

$= AXY + Y^*AY$.

Then one can deduce that :

$$\rho_A(XY) \neq X\rho_A(Y) + \rho_A(X)Y .$$

Now the following remark . shows the mapping ρ_A is also not * - a derivation .

Remark (3.5.2) :

Since $\rho_A(X + Y) = A(X + Y) + (X + Y)^*A$,

$= AX + AY + X^*A + Y^*A$,

$= AX + X^*A + AY + Y^*A$

$= \rho_A(X) + \rho_A(Y)$.

Now ,

$X\rho_A (X) \rho_A(X) X^* = X[AX + X^*A] X^*$,

$= XAX + XX^*A + AXX^* + X^* AX^*$,

So $\rho_A(X^2) = (AX^2 + (X^*)^2 A)$,

and $\rho_A(X^2) \neq X\rho_A(X) + \rho_A(X)X^*$.

then ρ_A is not* - a derivation

(3.6) On The Range of μ_A (X) :

In this section , we study and discuss the range $\mu_A(X)$, where

$$\mu_A = \mu_A(X) = AX + XA , \quad X \rfloor B(H)$$

Where A is a fixed operator in B(H) .

It is clear that the map μ_A is a linear map . Also , the map μ_A is bounded , since $\| \mu_A \| = \| AX + XA \| \leq M \| X \|$,

where $M = 2 \|A\| \geq 0$. Then μ_A is bounded .

The following steps shows that Range $(\mu_A)^* \neq$ Range (μ_A) .

Range $(\mu_A)^* = \{(AX + XA)^* , X \rfloor B(H)\}$,

$= \{A^*X^* + X^*A^* , X \rfloor B(H)\}$,

$\neq$ Range (μ_A) .

Also , α Rang $(\mu_A) = \{ \alpha (AX + XA) , X \rfloor B(H)\}$,

$= \{A(\alpha X) + (\alpha X) A , X \rfloor B(H)\}$,

Let $\alpha X = X_1$

$= \{AX_1 + X_1A , X_1 \rfloor B(H)\}$,

$=$ Range (μ_A) .

The following remark shows that mapping μ_A is not a derivation .

Remark (3.6.1) :

Since $\mu_A(XY) = A(XY) + (XY)A$.

For all $X\mu_A(Y) = X(AY) + X(YA)$,

$= XAY + XYA$.

Also , $\mu_A(X)Y = AXY + XAY$,

Then one can deduce that :

$$\mu_A(XY) \neq X\mu_A(Y) + \mu_A(X)Y .$$

Now , the following remark shows that the mapping μ_A is also not* - a derivation .

Remark (3.6.2) :

Since $\mu_A(X + Y) = A(X + Y) + (X + Y)A$,

$= AX + AX + XA + YA$,

$= (AX + XA) + (AY + YA)$,

$= \mu_A(X) + \mu_A(Y)$.

Now ,

$X\mu_A(X) + \mu_A(X)X^* = X[AX + XA] + [AX + XA]X^*$

$= XAX + X_2A + AXX^* + XAX^*$,

So , $\mu_A(X_2) = (AX_2 + X_2A)$,

and $\mu_A(X_2) \neq X\mu_A(X) + \mu_A(X)X^*$.

therefore μ_A is not * - a derivation .

Chapter Four

Generalization of the Lyapunov Equations

The operator equation of the form :

$$A^* x B x B x A = W , \ldots\ldots\ldots\ldots.. \quad (4\text{-}1)$$

Where A , B and W are given operator defined on H , X is the unknown operator that must be determined , and A* is the adjoint of A.

The above operators equation is one of the generalization continuous – time Lyapunov operator equation .

(4.2) : The Nature of The Solution for Generalization Lyapunov Equations :

Now , the nature of the solution for more general of the continuous – time Laypunov operator equation are studied for special types operators .

Proposition (4.2) :

If B and W are self – adjoint operators , and the operators equation (4.1) has only one solution X then this solution is also self – adjoint ,

Proof :

Consider to operator equation

$$A^*XB + BXA = W ,$$

$$(AXB + BXA)^* = W^* ,$$

$$A^*X^*B^* + B^*X^*A^* = W^*$$

$$A^*X^*B + BX^*A = W ,$$

Since X is a unique solution .

So X = X* , then is self – adjoint .

Proposition (4.2.2) :

If B is a skew – adjoint , W is a self – adjoint and A is any operator and if the equation (4.1) has only one solution then this solution is a skew – adjoint .

Proof :

Consider equation (4.1)

$$A^*XB + BXA = W ,$$

$$(A^* XB + BXA)^* = W^*$$

$$A^*X^*B^* + B^*X^*A = W^* .$$

$$A^*X^* (-B) + (- B)X^*A = W^* ,$$

$$A^* (- X^*) B + B (- X^*) A = W .$$

Since X is a unique solution , so $X = - X^*$

Then X is a skew – adjoint .

Proposition (4.2.3) :

If B and W are skew – adjoint operators , A is any operator and if the equation (4-1) has only one solution then this solution is self – adjoint .

Proof :

Consider equation (4.1) ,

$$A^* XB + BXA = W ,$$

$$- (A^* XB + BXA)^* = - W^*$$

$$- B^*X^*A - A^*X^*B^* = - W^*$$

$$A^*X^* (- B)^* + (- B^*) X^*A = - W^*$$

$$A^* X^*B + BX^* A = W .$$

Since X is a unique solution , then

$X^* = X$.

Proposition (4.2.4) :

If B and W are self – adjoint operators , A is any operator and if the equation (4.1) has only one solution and this solution is also self – adjoint .

Proof :

Consider equation (4.1) ,

$$A^*XB + BXA = W ,$$

$$(A^*XB + BXA)^* = W^* ,$$

$$A^* X^* B^* + B^*X^*A = W^* ,$$

$$A^* X^* B + BX^* A = W .$$

Since X is a unique , so $X^* = X$

The X is a self – adjoint .

Proposition (4.2.5) :

If B is a self – adjoint , W is a skew – adjoint , A is any operator and if the equation (4.1) has only one solution then this solution is a skew – adjoint .

Proof :

Consider (4.1) ,

$$A^*XB + BXA = W ,$$

$$- (A^* XB + BXA)^* = - W^*$$

$$A^* (- X^*) B^* + B^* (- X^*) A = - W^* ,$$

$$A^* (- X^*) B + B (- X^*) A = W .$$

Since equation (4.1) has only one solution , so X = - X* , the X is a skew – adjoint .

Remark (4.2.1) :

Consider equation (4.1) , If B is a self – adjoint , W is a self – adjoint and A is any operator , Then X is not necessary self – adjoint .

Remark (4.2.2) :

Consider equation (4.1) , If A and W are self – adjoint operators , B is any operator , Then X is not necessary self – adjoint .

Remark (4.2.3) :

If A and B are compact operators and the solution X of equation (4.1) exist , then this solution is not necessary compact .

The X is a self – adjoint .

Proposition (4.3.5) :

If B is a self – adjoint , W is a skew – adjoint , W is a skew – adjoint , and A is any operator and if the equation (4.4) has only solution and this solution is also self – adjoint .

Proof :

Consider equation (4.4) ,

$A^* x B + B x A = W$,

$- (A^* x B + B x A)^* = - W^*$,

$A^* (- X^*) B^* + B^* (- X^*) A = - W^*$,

$A^*(- X^*)^* B + B (- X^*)A = W$.

Since equation (4.4) has only one solution , so $X = - X^*$, Then X is a skew – adjoint .

Remark (4.3.1) :

Consider equation (4.4) , if

(1) B is a skew – adjoint , W is a self – adjoint , and A is any operator . Then X is not necessary self – adjoint .

(2) A and B are self – adjoint operators , B is any operator . The X is not necessary self – adjoint .

Remark (4.3.1) :

If A and B are compact operators and the solution of equation (4.4) exist , then this solution is not necessary compact .

Example (4.3.1) :

Consider the following operator equation

$A^* x B + B x A = A^* B + BA$,

Where A and B are compact operator , it is clear that X = I is a solution of the above operator equation , but I is not compact .

Now , we study that nature of the solution of the operator equations .

$(A^*)^2X + XA^2 + tAXA = W$, (4.9)

and $A^2X + XA^2 + tAXA = W$, (4.10)

the following proposition shows if A and W are self – adjoint operators and if equation (4.10) has a unique solution then this solution is self – adjoint .

Proposition (4.3.6) :

Consider equation (4.10) w , If A and W are self – adjoint operators then X is self – adjoint .

Proof :

Consider $A^2X + XA^2 + tA\,x\,A = W$, $t \in R$

$(A^2X + XA^2 + tA\,x\,A)^* = W^*$, $t \in R$

$X^*(A^*)^2\ X^* + tA^*X^*A^* - W$.

Since A and W are self – adjoint then $A = A^*$ and $W = W^*$.

So , $X^*\ A^2 + A^2X^* + tAX^*\ A = W$, $t \in R$

$A^2X^* + X^*A^2 + tAX^*A = W$.

Then X in self – adjoint .

Remark (4.3.3) :

Consider equation (4.10) , If A is self – adjoint operator and W is any operator then solution X is not necessary self – adjoint operator .

The following example the above remark .

Example (4.3.2) :

Consider equation (4.10) ,take A = A* = $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, W = $\begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$

Any operator , and t = 2

Equation (4.10) becomes :

$A^2X + XA^2 + 2AXA = W$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} + \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + 2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 & X_2 \\ X_3 & X_4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 3 & 1 \end{bmatrix}$$

After simple computation , the solution of equation in case take the form :

X = $\begin{bmatrix} \frac{1}{2} & 0 \\ \frac{3}{4} & 0 \end{bmatrix}$ ≠ X* , where α any arbitrary number .

<u>Remark (4.3.4) :</u>

Consider (4.9) ,

(1) If A is is self – adjoint operator then the solution X is not necessary self = adjoint .

(2) if W is self – adjoint operator then the solution X is not necessarily self – adjoint .

To explain the above remarks see example (4.3.2) .

<u>Remark (4.3.5) :</u>

Consider equation (4.9) , if A and W are normal operators , them the solution X is not necessary exists .

The following example explain this fact .

Example (4.3.3) :

Consider equation (4.9) , A = $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, b ≠ 0 .

Notes that A ≠ A* in general .

A normal , but not self – adjoint .

Take

A = $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, W = $\begin{bmatrix} 1 & -3 \\ 3 & 1 \end{bmatrix}$ and t = 2 .

After simple computations , we get

$-2X_2 - 2X_4 = 1$ and $-2X_2 + 2X_3 = -3$

$-2X_4 - 2X_1 = 1$ and $-2X_3 - 2X_2 = 3$

Has no solution .

The question now is pertinent , does equation (4.9) has a normal solution . To answer this question consider the following example :

Example (1.3.4) :

Consider equation (4-9) and take W = 0 , equation (4-9) becomes :

$(A^*)^2X + XA^2 + tAXA = 0$

It is Cleary X = 0 and 0 is a normal operator .

Now . We study the nature of the solution of equation (4.9) and equation (4.10) for compact operator .

The following proposition shows if the operator A is compact then the operator W is also compact .

Proposition (4.3.7) , [21]

Consider equation (4.10) , if A is compact operator then W is compact operator .

Proof :

Equation (4.10) , can be written as

AAX + XAA + tAXA = W , since A is compact operator then AA is compact (i.e. A^2 compact) . So A^2X is compact . A is compact operator then XA is compact and XAA is compact , so XA^2 is compact . AX is compact then AXA is compact and t AXA is compact (t any scalar) .

So $A^2X + XA^2 + tAXA$ is compact , then W is compact operator .

Remark (4.3.6) :

Consider equation (4.10) , if A and W are compact operators then the solution X is not necessarily compact operator .

The following example explain to above remark .

Example (4.3.5)

Consider the following operator equation :

$A^2X + XA^2 + tAXA = (2 + t)A^2$, where t is any scalar .

It is clear that X = I is a solution of the above operator equation , but I is not compact operator .

Now , we study and discuss the nature of the solution of equation (4.6) and eq. (4.8) for special types of operators .

Proposition (4.3.8) :

If B and W are self – adjoint operators , A is any operator and if the equation (4.6) has only one solution , then this solution is self – adjoint .

Proof :

Consider equation (4.6) ,

$A^*BX + XBA = W$,

$(A^*BX + XBA)^* = W^*$,

$X^* B^* A^* + A^* B^* X^* = W^*$.

Since B and W are self – adjoint ,

Then A* BX* + X* BA = W .

Since equation (4.6) has only solution , so X* = X , then X is self – adjoint .

Proposition (4.3.9) :

If B and W are self – adjoint operators , W is a skew – adjoint , and A is any operators and if the equation (4.6) has only one solution , then this solution is skew – adjoint .

Proof :

Consider equation (4.6) ,

A*BX + XBA = W ,

- (A* BX + XBA)* = W* ,

- X*B*A + A* B*X* = - W* ,

A* B* (- X*) + (- X*) B*A = - W* ,

A* B (- X*) + (- X*) BA = W .

Since equation *4.6) has only one solution , so X = - X* , then X is a skew – adjoint .

Proposition (4.3.10) :

If B is a skew – adjoint operator , W is a self – adjoint operator , A is any operator .

Equation (4.6) has only one solution , then X is a skew – adjoint .

Proof :

Consider equation (4.6) ,

$A^* BX + XBA = W$,

$(A^* BX + XBA)^*$,

$X^*B^*A + A^*B^*X^* = W^*$.

Since B is a skew – adjoint , then

$A^*B (-X) + (- X^*) BA = W$.

Since equation (4.6) has one solution , so $X = -X^*$, then X is skew – adjoint .

Remark : (4.3.7) :

Consider equation (4.6) , If A and W are self – adjoint operator , b is any operator . Then X is not necessarily self – adjoint .

Remark (4.3.8) :

If A and B compact operators and the solution of equation (4.6) exist , them it solution is not necessarily compact .

this fact can easily be seen in example (4.3.1) .

Chapter Five

The Range of The Generalization Lyapunov Operator Equation

The Range of The Generalization Lyapunov Operator Equations

In This Chapter , we study the range of the generalization of continuous – time Lyapunov operator equations .

(5.1) : The Rang of τ_{AB} :

In this section , we study and discuss the properties of τ_{AB} .
Recall that , a linear mapping τ from a ring R it self is called a derivation , if $\tau(ab) = a\tau(a)^{b}$, for all a , b in R , [8] .

Define the mapping $\tau = B(H) \rightarrow B(H)$ by

$\tau(X) = \tau_{AB}(X) = A^*XB + BXA$, X ⌋ B(H) .

Where A and B are fixed operators in B (H) .

It is clear that the map τ_{AB} is a linear map ,

in fact

$\tau_{AB}(\alpha X_1 + \beta X_2) = A^* (\alpha X_1 + \beta X_2) B + B (\alpha X_1 + \beta X_2) A$

$= \alpha A^*X_1B + \beta A^*X_2B + \alpha BX_1A + \beta BX_2A$

$= \alpha (A^*X_1B + BX_1A) + \beta(A^*X_2B + BX_2A)$

$= \alpha\tau_{AB} (X_1) + \beta\tau_{AB} (X_2)$.

Also , the map τ_{AB} is bounded , since :

$\| \tau_{AB} \| = \|A^*XB + BXA\| \leq \|A^*XB\| + \|BXA\|$

$\leq \|X\|[2\|AB\|]$,

thus $\| \tau_{AB} \| \leq M \|X\|$, where $M = 2 \|AB\|$, So, τ_{AB} is bounded.

The following remark shows that the mapping τ_{AB} is not a derivation.

Remark (5.1.1):-

Since $\tau_{AB}(XY) = A^*(XY)B + B(XY)A$,

and $X\ \tau_{AB}\ (Y)=X.A^*YB+XBYA$.

Also, $\tau_{AB}(X)Y=A^*XBY+BXAY$.

Then we can deduce that

$$\tau_{AB}(XY) \neq X\,\tau_{AB}(Y) + \tau_{AB}(X)\,Y$$

Proposition (5.1.1):-

(1)Range $(\tau_{AB})^*$ = Range (τ_{AB}), if B is self— adjoint

(2) α Range (τ_{AB}) = Range (τ_{AB})

Proof:-

(1) Since Range $(\tau_{AB})^* = \{(A^*XB+BXA)^*, X \rfloor B(H)\}$.

Then Range $(\tau_{AB})^* = \{(XB)^*A + A^*(BX)^*, X \rfloor B(H)\}$, =

$\{B^*X^*A + A^*X^*B^*, X \rfloor B(H)\}$,

$= \{A^*\text{-}X^*{}_{.}B^* + B^*X^*A,. X \rfloor B(H)\}$.

Since B is self - adjoint $(B = B^*)$, then

Range $(\tau_{AB})^* = \{A^* X^* B + B X^* A, X ⌋ B (H)\}$,

$= \{A^* X_1 B + BX_1 A, X_1 ⌋ B (H)\}$,

where $X^* = X_1$ Therefore, Range $(\tau_{AB})^*$ = Range (τ_{AB}).

(2) α Range $(\tau_{AB}) = \{\alpha (A^*XB + BXA) , X ⌋ B (H)\}$,

$= \{ A^*(\alpha X)B + B(^{a} X)A , X ⌋ B (H)\}$,

$= \{ A^*X_1B +BX_IA , X_1 ⌋ B (H)\}$,

where X, = α X, then

α Range (τ_{AB}) = Range (τ_{AB}).

The following remark shows that the mapping τ_{AB} is not * - derivation.

Remark (5.1.2):-

Since $\tau_{AB} (X + Y) = A^* (X + Y) B + B (X — Y) A$

$=A^*XB+A^*YB+BXA+BYA$

$= A^*XB + BXA2 + A^*YB + BYA$

$= \tau_{AB} (X) + \tau_{AB} (Y)$,

Now, $X \tau_{AB} (X) + \tau_{AB} (X) X^*$

$= X (A^* XB+BX A) + (A^* XB+B X A) X^*$.

So, $(X^2) = A^*X^2 B +B X^2 A$,

then , $\tau_{AB} (X^2) \neq X^{\tau AB} (X)^{\tau AB} (X) X^*$.

(5:2): The Range of τ_A :-

In this section, we study and discuss the properties of τ_A

Define the mapping $\tau : B(H) \rightarrow B(H)$ by:

$$\tau(X) = \tau_A(X) = (A^*)^2 X + XA^2 + tAXA \text{ , } X \rfloor B(H).$$

Where A is a fixed operator in B (H) and t is any scalar.

It is clear that the map τ_A is a linear map. Also, the map τ_A is bounded, since

$$\|\tau_A\| = \|(A^*)^2 X + XA^2 + tAXA\| \leq \|(A^*)^2 X\| + \|XA^2\| + \|t\|AXA\|$$

$$\leq \|(A^*)^2\|\|X\| + \|X\|\|A^2\| + \|t\|\|A\|\|X\|\|A\|$$

$$\leq \|A^2\|\|X\| + \|X\|\|A^2\| + \| + \|t\|\|X\|\|A^2\|$$

$$\leq (2 + \|t\|)\|A\|^2 \|X\| .$$

Let $\quad M = (2 + |t|)\|A\|^2 \geq 0$.

SO $\quad \|(A^*)^2 X + XA^2 + tAXA\| \leq M\|X\|$.

then τ_A is bounded .

The following steps show (Range (τ_A))* $\neq$ Range (τ_A)

$$(\text{Range}(\tau_A))^* = \{((A^*)^2 X + XA^2 + t A X A)^*, X \rfloor B(H)\} ,$$

$$= \{X^*((A^*)^2)^* + (A^2)^* X^* t (XA)^* A^*, X \rfloor B(H)\},$$

$$= \{X^*A^2 + (A^*)^2 X^* + tA^* X^* A^*, X \rfloor B(H)\}.$$

Let X1 =X*

$\{(A^*)^{2.}$ Xi +X1 A^2 + t A* X1 A*, X1 $\rfloor$ B (H)} $\neq$ Range (τ_A)

Also, α Range (τ_A)= α (($A^*)^2$ X+XA^2 + t A X A), X $\rfloor$ B (H)},

= {α ($A^*)^2$ X+ α X A^2 + t A (α X) A, X $\rfloor$ B (H)},

= {($A^*)^2$ (α X) + (α X) A^2 + t A (α X) A, X $\rfloor$ B (H)).

Let X1 = αX

= {($A^*)^2$ X1 + X1 A^2 +t A XI A, X1 ^{E} B (H)}

= Range (τ_A).

The following remark shows the mapping τ_A is not derivation.

<u>Remark (15.2.1) :</u>

Since τ_A (X Y) = ($A^*)^2$ (XY) + (XY) A^2 + τ_A (X Y) A, for all X, Y $\rfloor$ B (H) and

X τ_A A (Y) = X $(A)^2$Y + X Y A^2+ X t A YA.

Also , τ_A (X) Y = (($A^*)^2$ X +XA^2 + tA XA)Y

= ($A^*)^2$ X Y + XA^2Y + t AX AY.

Then we can deduce that τ_A (XY) $\neq$ X τ_A (Y) + τ_A (X) Y.

Now, the following remark shows the mapping τ_A is also not * - derivation.

Remark (5.2.2):-

Since $\tau_A (X+Y) = (A^*)^2 (X+Y) + (X+Y)A^2 + tA(X+Y) A$

$= (A^*)^2X + (A^*)^2 Y + XA^2 + Y A^2 + tAXA + tAYA$

$= (A^*)^2 X + XA^2 + tAXA + (A^*)^2 Y + YA^2 + tAYA$

$= \tau_A (X) + \tau_A (Y).$

Now,

$X \tau_A (X) + \tau_A (X) X^* = X (A^*)^2 X + X^2 A^2 + tXAXA + (A^*)2XX^* + X A^2 X^* t AXAX^*.$

So, $\tau_A (X^2) \neq X \tau_A (X) + \tau_A (X) X^*$

Then τ_A is not * - derivation,

(5.3) : The Range of τ_{tA}

In this section, we study and discuss the range of τ_{tA} , where

$$\tau\ (X) = \tau_{tA}\ (X) = A^*\ X + t\ X\ A,\ X \rfloor (H)\ ,$$

where A is a fixed operator in B (H), t is any scalar. The map τ_{tA} is a linear map.

$$\tau_{tA}\ (\alpha X_1 + \beta X_2) = A^*\ (\alpha X_1 + \beta X_2) + t(\alpha X_1 + \beta X_2)A$$

$$= \alpha A X_1 + \beta A^* X_2 + t\alpha X_1 A + t\beta X_2 A$$

$$= \alpha A^*\ X_1 + \alpha t X_1 A + \beta A^* X_2 + \beta t X_2 A$$

$$= \alpha\ (A^* X_1 + t X_1 A) + \beta(A^* X_2 + t X_2 A)$$

$$= \alpha\tau_{tA}(X_1) + \beta\tau_{tA}(X_2)\ .$$

Also, the map τ_{tA} is bounded since,

$$\|\ \tau_{tA}\ \| = \|A^*\ X + tXA\ \| \leq \|A^*\ X\| + \|t\|\|XA\|$$

$$\leq \|\ X\ \|\ [(\|A^*\ \| + \|t\|)\ \|A\|]\ .$$

But A $\rfloor$B (H) and $\|A^*\| = \|A\|$, thus $\|\ \tau_{tA}\| \leq M\ \|X\|$, where

$M = (1 + |t|)\|A\|$, so τ_{tA} is bounded.

The following remark shows that the mapping τ_{tA} is not derivation.

Remark (5.3.1):-

Since τ_{tA} (XY) = A* (XY) t (XY) A, for all X, Y $\rfloor$B (H)

and X τ_{tA} (Y) = X A* Y + t XYA.

Also, τ_{tA} (X)Y = A* XY + t XYA.

Then, one can deduce that τ_{tA} (XY) $\neq$ X τ_{tA} (Y) + τ_{tA} (X)Y.

it is easily seen that the mapping τ_{tA} is not * - derivation.

Remark (5.3.2):-

Range $(\tau_{tA})^* \neq$ Range (τ_{tA}) .

Now, we have the following proposition.

Proposition (5.3.1) :-

α Range (τ_{tA}) = Range (τ_{tA}).

Proof:-

α Range (τ_{tA}) = $\{\alpha(A^*X + t\,X\,A),\ X \rfloor B(H)\}$,

$= \{A^*(\alpha X) + t(\alpha X)A,\ X \rfloor B(H)\}$.

Let $X_1 = \alpha X$, then:

α Range (τ_{tA}) = $\{A^* X_1 + t\,X_1\,A,\ X_1 \rfloor B(H)\}$

= Range (τ_{tA}) .

(5.4): The Range of ρ_{AB} :

Define the mapping $\rho : B(H) \to B(H)$ by:

$p(X) = \rho_{AB}(X) = A^*\ BX + XBA$, $X \rfloor B(H)$

Where A and B are fixed operators in B (H), and A* is the adjoint of A.

The map ρ_{AB} is a linear,

in fact $\rho_{AB}(\alpha X_1 + \beta X_2)\quad + A^*B(\alpha X_1 + \beta X_2) + (\alpha X_1 + \beta X_2)BA$

$= \alpha A^*BX_1 + \beta A^*BX_2 + \alpha X_1BA + \beta X_2BA$

$= \alpha(A^*BX_1 + X_1BA) + \beta(A^*BX_2 + X_2BA)$

$= \alpha\,\rho_{AB}(X_1) + \beta\,\rho_{AB}(X_2)$.

Also, the map ρ_{AB} is bounded, since:

$$\|\rho_{AB}\| = \|A^*BX + XBA\| \leq \|A^* BX\| + \|XBA\|$$

$$\leq \|X\|[2\|AB\|] ,$$

thus, $\|\rho_{AB}\| \leq M \|X\|$,where $M= 2 \|AB\|$,

so, $\|\rho_{AB}\|$ is bounded.

The following remark shows that the mapping ρ_{AB} is not derivation.

Remark (5.4.1):-

Since $\rho_{AB} (XY) = A^* B (XY) + (XY)$-I3A, $\forall$ X, Y ⌋B (H)

and $X \rho_{AB} (Y) = XA^*B(Y) + XYBA$.

Also, $\rho_{AB} (X) Y = A^* BXY + XBAY$.

Then we can get that:

$$\rho_{AB} (XY) \neq X \rho_{AB} (Y) + \rho_{AB} (X) Y.$$

Proposition (5.4.1) :

(1) Range $(\rho_{AB})^*$ = Range (ρ_{AB}), if B is a self-adjoint operator.

(2) α Range (ρ_{AB})= Range (ρ_{AB}).

Proof:-

(1)Since Range $(\rho_{AB})^* = \{(A^* BX + XBA)^*, X \rfloor B(H)\}$,

$= \{A^* B^* X^* + X^* B^* A, X \rfloor B(H)\}$.

Since B is a self - adjoint operator, then:

Range $(\rho_{AB})^* = \{A^* B X^* + X^* BA, X \rfloor B (H)\}$

$= \{A^* B X1 + X1 BA, X1 \rfloor B(H)\}$,

where X1= X*. Therefore, Range $(\rho_{AB})^*$ = Range (ρ_{AB})

(2) α Range $(\rho_{AB}) = \{\alpha (A^*BX + XBA) : X \rfloor B(H))$,

$= \{A^*B (\alpha X) + (\alpha X) BA: X \rfloor B(H)\}$,

$= \{A^*B X1 + X1 BA: X1 \rfloor B (H)\}$,

where X1 = αX, then :

α Range (ρ_{AB}) = Range (ρ_{AB}).

Remark (5.4.2):-

The mapping ρ_{AB} is not * - derivation.

(5.5): The Range of J_{AB}:-

Define the mapping J: B (H) → B (H) by:

$$J(X) = J_{AB}(X) = BAX + XAB, \; X \rfloor B(H) \,,$$

where A and B are fixed operators in B(H).

The map J_{AB} is linear.

Also, the map J_{AB} is bounded, Since $\|J_{AB}\| \leq M\|X\|$, where

$M = 2\|AB\|$.

Remark (5.5.1):-

$$J_{AB}(XY) \neq J_{AB}(Y) + J_{AB}(X)\, Y.$$

According to above remark, the mapping J_{AB} is not a derivation.

Proposition (2.5.1):-

(1) Range $(J_{AB})^* \neq$ Range (J_{AB}).

(2) α Range (J_{AB}) = Range (J_{AB})

Proposition (5.5.2):-

If A and B are self - adjoint operators, then:

$$\text{Range}\,(J_{AB})^* = \text{Range}\,(J_{AB}).$$

The proof of above propositions directly from definitions:

Remark (5.5.2):-

$$J_{AB}(X^2) \# XJ_{AB}(X) + J_{AB}(X) X^*$$

According to the above remark, the mapping is not * - derivation.

(5.6): The Spectrum of τ_{AB}

In this section, we study the relation between the spectra of L_{A*} R_B RI_B L_A with the spectra of A* , B and A respectively.

Where $L_A{}^*(X)^{\cdot}$ A*X , $R_B(X)$=XB , $R1_B(X)$=BX , and $L_A(X)$ =XA

Let B(B(H)) be the Banach algebra of operators of B(H) considered as a Banach space.

Definition (5.6.1), [11] :

Let X be a Banach space over € , and let T ⌋ B(H), define :

$$\sigma_\pi(T) = \{\lambda ⌋ € ;\ T - \lambda I \text{ is not bounded below } \}$$

$\sigma_\pi(T)$ is called the approximate point spectrum of T.

An important subset of $\sigma_\pi(T)$ is the point spectrum or eigen values X of T which we denoted by $\sigma_p(T)$ where,

$\sigma_\pi(T) = \{ \lambda ⌋ € : Ker (T- \lambda I) \quad \neq \{0\} \}$.

Also, we define $\sigma_\pi(T) =\{ \lambda ⌋ € ; T- \lambda I$ is not subjective$\}$,

$\sigma_\pi(T)$ is called the defect spectrum of T

<u>Notation (5.6.1) , [21] :</u>

For A , B G B(H) , X is any Banach space , Let

1. $\sigma (A)+ \sigma (B) = \{\alpha + \beta : \alpha ⌋ \sigma (A), \beta ⌋ \sigma (B) \}$,
2. $\sigma_\pi (A)+ \sigma_\pi (B) = \{\alpha + \beta : \alpha ⌋ \sigma_\pi (A), \beta ⌋ \sigma_\pi (B) \}$,
3. $\sigma_\pi (A)+ \sigma_\pi (B) = \{\alpha\beta : \alpha ⌋ \sigma_\pi (A), \beta ⌋ \sigma_\pi (B) \}$,
4. $\sigma_\delta (A)+ \sigma_\delta (B) = \{\alpha + \beta : \alpha ⌋ \sigma_\delta (A), \beta ⌋ \sigma_\delta (B) \}$,
5. $\sigma_\delta (A)+ \sigma_\delta (B) = \{\alpha\beta : \alpha ⌋ \sigma_\delta (A), \beta ⌋ \sigma_\delta (B) \}$.

In the following theorem we give the relation between the parts of spectrum of the sum of two operator A and. B define on a Banach space X and the stun of the spectrum .

Theorem (5.6.1),[22] :-

If $A , B \rfloor B(H)$, and $AB = BA$, then

(i) $\sigma_\pi (A + B) \div \sigma_\pi (A) + \sigma_\pi (B)$

(ii) $\sigma_\pi (AB) \div \sigma_\pi (A) \sigma_\pi (B)$.

Corollary (5.6.1),[23]:-

If $A , B \rfloor B(H)$ and $AB = BA$ then :

(i) $\sigma_\pi (A+ B) \div \sigma_\pi (A) + \sigma_\pi (B)$

(ii) $\sigma_\pi (AB) \div \sigma_\pi (A) \sigma_\pi (B)$.

In [22] , Herro proved that if X is a Hilbert space H then theorem (5.64) and corollary (5.6.1) become

Remark (5.6.1) :-

If $A , B \rfloor B(H)$ and $AB = BA$ then

1. $\sigma_\pi (A + B) = \sigma_\pi (A) + \sigma_\pi(B)$ and $\sigma_\pi (AB) = \sigma_\pi (A) \sigma_\pi(B)$.
2. $\sigma_\pi (A +B) = \sigma_\pi (A) + \sigma_\pi (B)$ and $\sigma_\pi (AB) = \sigma_\pi (A) \sigma_\pi (B)$.

Now, define the mapping L and R from B(H) into B(B(H)) Such that

$$\left.\begin{array}{l} L_A(X) = AX \\ R_B(X) = XB \end{array}\right\} \quad A ,B \rfloor B(H) .$$

Now we return to our problem, we want to relate the spectra of L_A and R_B with the spectra of A and B , respectively.

<u>Lemma (5.6.1), [23] :</u>

Let A , B ⌋ B(H) , then :

1. $\sigma_\pi (L_A) = \sigma_\pi (A) , \sigma_\delta (R_B) + \sigma_\pi (B)$.
2. $\sigma_\delta (L_A) = \sigma_\delta (A) , \sigma_\pi (R_B) + \sigma_\delta (B)$.

According to above properties ,consider the following

<u>Corollary (5.6.2)</u>

1. $\sigma_\pi (\tau_{AB}) = \sigma_\pi (A^*) \sigma_\pi (B) - \sigma_\delta (B) \sigma_\delta (A)$.
2. $\sigma_\delta (\tau_{AB}) = \sigma_\delta (A^*) \sigma_\delta (B) - \sigma_\pi (B) \sigma_\pi (A)$.

References:

11]. Al - Ganabi, E.H, The numerical range and normal operators, M.sc. thesis submitted to the college of science - university of Baghdad, 1996,

[2J. Amina R.A. On the range of Jordan * derivations, M.sc. thesis, college of science university of Baghdad, 1999.

[3];,Bhatia R.,A note on the Lapunov equation, linear algebra and its applications vol.259, pp(7l76), 1997.

[4]. Bhatia R., and Sner,L. Positive linear maps and the Lyapunov equation, operator theory: advances aid applications• vol.130, pp(107-i20), 2001.

[5]. Bhtia R. and Rosenthal, P. How and why to solve the operator equation AX - XB = Y, Bull - London Math.s'oc., vol.29 , pp(l-l2), 1997

[6]. Bhatia. R. Davis C. and kosis P. An external proplem in Fourier analysis with applications to operator theory , J. Functioflal Anal. Vol.82 , pp(l38-1 50), 1989.

[7]. Berberian S.K.. Introduction to Hi.lhert s$ce, Oxford university tress, inc. New-York, 1961.

[8]. Campbel, S.L. Linear operators for which ^{T*T} and T Commute, proc. Amer. Math. Soc. Vol.34, 1974.

[9]. Embry, M.R. Similarities involving normal operators.on Hubert space, Pacific. J. Math,vol.35, 1970.

[10]. Embiy, M.R. ^{Nth} root of operators. Proc. Amer. Math. Soc.

Vol.19, 1968.
[11]. Halmos, P.R. A Hilbert space problem book, Springer - erlag, New-York, inc. 1982.
[121. Lee,S;H. and Ryoo; ..S.:. Some properties of certain nonhyponormal operators, Bull-Korean, Math: goc. Vol 31 1994. -
[13]. K.UfE, E.A. About the solutions of Lyapunov equations, ph .D.thesis, Al-a\lahrain urns ersity, 2005 .
[14]Bliatia, R. and Parthasarathy K. R., "Positive Definite Functions and Operator Inequalities", Bull - London Math. Soc., Vol. 32, pp. 214 - 288,(2000).
[1 5]Dragon, S. Djordjevic, "Explicit Solution of the Operator Equation A* X + X*· A = B", Republic Serbia, (2005).
[16]Davis, C. and Rosenthal. ," Solving linear operator equation ", Canadian J.Math., XXVI ,(1974) ,1384-1389.
[17]Emacl, A. "The Nature of The Solution for The Generalization Lyapunov Equations", J. Al-Qadisiah, No. 4, Vol. 11, (2006).
[18] Frank, A., "Matrices", MC Grow ,- Hill, Publishing Company LTD., (1962). {I 9Herro, D.A, 'Approximation of Hilbert space operators ", Vol. 1,Pitinan Advanced Publishing Program, (1982) .
[20] Goldslen, J. A., "On The Operator Equation AX XB Q", proc. Amer. Math. Soc., Vol. 70, pp. 31 - 24, (1978),
[21]Bartels, R. H. and Stewart, G. W., "Algorithm 432: Solution of the Matrix Equation AX + XB C". Comm. ACM, Vol, 15, pp. 820 - 826, (1972).

[22]Schweinsbreg,¸A:, "The Operator Equation AX - XB = C with Normal A and B", Pacified of Math., Vol. 102, pp. 445 - 453, (1982).

[23]Lin, S. C. and Shaw, S. Y., "On Equations AX = Q and SX - XT = Q", J. Funct. Anal., Vol. 77, pp. 352 - 262, (1988).

الخلاصة :

الهدف الرئيس من هذا العمل يمكن تقسيمه إلى ثلاثة محاور رئيسية :

المحور الأول – قمنا بتحسين بعض المبرهنات لإثبات وجود وحدانية الحل لمعادلات ليبانوف أو شبه ليبانوف المؤثرة (المستمرة) .

المحور الثاني – دراسة حل معادلة ليبانوف المؤثرة لأنواع خاصة من المؤثرات ، وكذلك دراسة ومناقشة وجود وحدانية الحل لمعادلات ليبانوف وسلفستر المتقطعة المؤثرة لأنواع خاصة من المؤثرات بالإضافة إلى دراسة المدى .

المحور الثالث – هو دراسة المدى لمعادلات شبه لينانوف لأنواع خاصة من المؤثرات .

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وهي جزء من متطلبات نيل درجة فلسفة الدكتوراه في علوم الرياضيات

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